PUTNAM PRACTICE SET 10

PROF. DRAGOS GHIOCA

Problem 1. Prove that for each positive integer n, we have

$$\left(\frac{2n-1}{e}\right)^{\frac{2n-1}{2}} < \prod_{i=1}^{n}(2i-1) < \left(\frac{2n+1}{e}\right)^{\frac{2n+1}{2}},$$

where e is base of the natural logarithm.

Problem 2. For any square matrix A with real entries, we can define

$$\sin(A) := \sum_{n=0}^{\infty} \frac{(-1)^n A^{2n+1}}{(2n+1)!},$$

i.e., the above series converges. Determine with proof whether there exists some matrix A with real entries such that

$$\sin(A) = \left(\begin{array}{cc} 1 & 2019 \\ 0 & 1 \end{array}\right).$$

Problem 3. Let $P \in \mathbb{R}[x]$ with the property that $P(x) \geq 0$ for all $x \in \mathbb{R}$. Prove that there exist polynomials $Q_1, Q_2 \in \mathbb{R}[x]$ such that $P(x) = Q_1(x)^2 + Q_2(x)^2$.

Problem 4. Let a_n be real numbers so that the following power series expansion holds:

$$\frac{1}{1-2x-x^2} = \sum_{n=0}^{\infty} a_n x^n.$$

Prove that for each integer $n \geq 0$, there exists a positive integer m such that $a_{n+1}^2 + a_n^2 = a_m$.